

# Embedded surfaces with Anosov geodesic flows, approximating spherical billiards

Mickaël Kourganoff\*

## Abstract

We consider a billiard in the sphere  $\mathbb{S}^2$  with circular obstacles, and give a sufficient condition for its flow to be uniformly hyperbolic. We show that the billiard flow in this case is approximated by an Anosov geodesic flow on a surface in the ambient space  $\mathbb{S}^3$ . As an application, we show that every orientable surface of genus at least 11 admits an isometric embedding into  $\mathbb{S}^3$  (equipped with the standard metric) such that its geodesic flow is Anosov. Finally, we explain why this construction cannot provide examples of isometric embeddings of surfaces in the Euclidean  $\mathbb{R}^3$  with Anosov geodesic flows.

## 1 Introduction

### 1.1 Anosov geodesic flows for embedded surfaces

The geodesic flow of any Riemannian surface whose curvature is negative everywhere is Anosov: in particular, any orientable surface of genus at least 2 can be endowed with a hyperbolic metric, for which the geodesic flow is Anosov. On the contrary, there is no Riemannian metric on the torus or the sphere with an Anosov geodesic flow.

If a closed surface admits an isometric embedding in  $\mathbb{R}^3$ , then it needs to have positive curvature somewhere. However, it is still possible to obtain an Anosov geodesic flow for such a surface, as shown by Donnay and Pugh [DP03]. More precisely, they showed that there exists a genus  $g_0$  such that for all  $g \geq g_0$ , the orientable surface of genus  $g$  admits an isometric embedding in  $\mathbb{R}^3$  whose geodesic flow is Anosov (see [DP04]). The value of  $g_0$  is completely unknown: in particular, it is unknown whether it is possible to embed isometrically a surface of genus smaller than one million in  $\mathbb{R}^3$  so that its geodesic flow is Anosov.

The same question may be asked for embeddings in the sphere  $\mathbb{S}^3$  endowed with the standard metric. Here, we have the following situation:

**Proposition 1.1.** *Any closed surface  $M$  isometrically embedded in  $\mathbb{S}^3$  admits at least one point at which the Gauss curvature is at least 1 (except if  $M$  is a torus or a Klein bottle).*

---

\*Institut Fourier, Grenoble University, France

*Proof.* The curvature of the surface is given at each point by  $K = k_1 k_2 + 1$ , where  $k_1$  and  $k_2$  are the principal curvatures of the surface. If  $K < 1$  everywhere, then the principal curvatures are nonzero and have different signs at each point. Thus the principal directions induce two nonsingular vector fields on  $M$ , which implies that the Euler characteristic of  $M$  must be zero.  $\square$

We will show for the first time that it is possible to embed isometrically in  $\mathbb{S}^3$  a Riemannian surface with Anosov geodesic flow. More precisely:

**Theorem 1.2.** *Every orientable surface of genus at least 11 admits an isometric embedding into  $\mathbb{S}^3$  such that its geodesic flow is Anosov.*

## 1.2 Approximating billiards by flattened surfaces

Birkhoff [Bir27] seems to be the first to suggest that billiards could be approximated by geodesic flows on flattened surfaces: he took the example of an ellipsoid whose vertical axis tends to zero, which converges to the billiard in an ellipse. Later, Arnold [Arn63] writes that smooth Sinai billiards could be approximated by surfaces with nonpositive curvature, with Anosov geodesic flows. In [Kou16a], we proved this fact for a large class of flattened surfaces.

In this paper, we prove a similar result for another class of objects. Consider a finite family of open disks  $\Delta_i$ , whose closures are disjoint, which have radii  $r_i < \pi/2$ , on the sphere  $\mathbb{S}^2$ : we will say that the billiard  $D = \mathbb{S}^2 \setminus \bigcup_i \Delta_i$  is a *spherical billiard with circular obstacles* (Figure 1). Define the billiard flow in the following way: outside the obstacles, the particle follows the geodesics of the sphere with unit speed; when the particle hits an obstacle, it bounces, following the usual billiard reflection law.

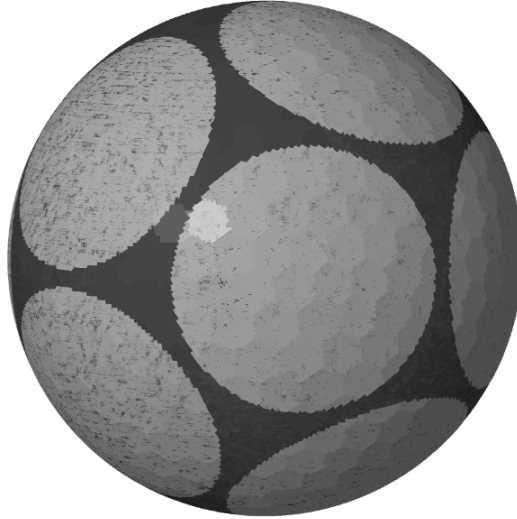


Figure 1: A spherical billiard with 12 obstacles.

The *horizon*  $H$  of a spherical billiard is the length of the longest geodesic of  $\mathbb{S}^2$  contained in the billard  $D$ .

The phase space  $\Omega$  of the billiard is defined as  $\Omega = T^1(\text{Int}(D))$  (the unit tangent bundle of the interior of  $D$ ). To define uniform hyperbolicity, we need to consider the set  $\tilde{\Omega}$  of all  $(x, v) \in T^1(\text{Int}(D))$  such that the trajectory starting at  $(x, v)$  does not contain any grazing collision (that is, each obstacle is reached with a nonzero angle).

We will use a definition of “uniformly hyperbolic billiard” which can be found in [CM06]:

**Definition 1.3.** The billiard flow  $\phi^t$  is *uniformly hyperbolic* if at each point  $x \in \tilde{\Omega}$ , there exists a decomposition of  $T_x\Omega$ , stable under the flow,

$$T_x\Omega = E_x^0 \oplus E_x^u \oplus E_x^s$$

where  $E_x^0 = \mathbb{R} \frac{d}{dt} \Big|_{t=0} \phi^t(x)$ , such that

$$\|D\phi_x^t|_{E_x^s}\| \leq a\lambda^t, \quad \|D\phi_x^{-t}|_{E_x^u}\| \leq a\lambda^t$$

(for some  $a > 0$  and  $\lambda \in (0, 1)$ , which do not depend on  $x$ ).

Smooth flat billiards with negative curvature and finite horizon are known to be uniformly hyperbolic [Sin70], but it is not the case for spherical billiards, as the following example shows:

**Example 1.4.** Consider six disjoint disks on the sphere, with the same radius  $r$ , whose centers are the vertices of a regular octahedron which is inscribed in the sphere. If the radius  $r$  is large enough, the billiard has finite horizon. However, there is a family of billiard trajectories which are parallel to the geodesic which is drawn on Figure 2, and thus, the billiard is not uniformly hyperbolic.

However, we will show that there exists a spherical billiard with 12 circular obstacles which is uniformly hyperbolic (see Figure 1).

### 1.3 Approximation by a closed surface

In [Kou16a], we have shown that, under some conditions, uniformly hyperbolic billiards may be approximated by smooth surfaces whose geodesic flow is Anosov. The main result of [Kou16a] only applies to flat billiards, but we will show in this paper that it is possible to approximate our spherical billiard by a surface in the ambient space  $\mathbb{S}^3$  such that the geodesic flow is Anosov (Theorem 2.2): see Figure 3.

It is tempting to try the same construction in the ambient space is  $\mathbb{R}^3$ : however, we will see that in this framework, the geodesic flow of a surface which approximates a spherical billiard is *never* Anosov (see Theorem 2.4). This result, which might seem surprising at first sight, is due to the accumulation of a high quantity of *positive curvature* beside the negative curvature which appears near the boundary of the billiard.

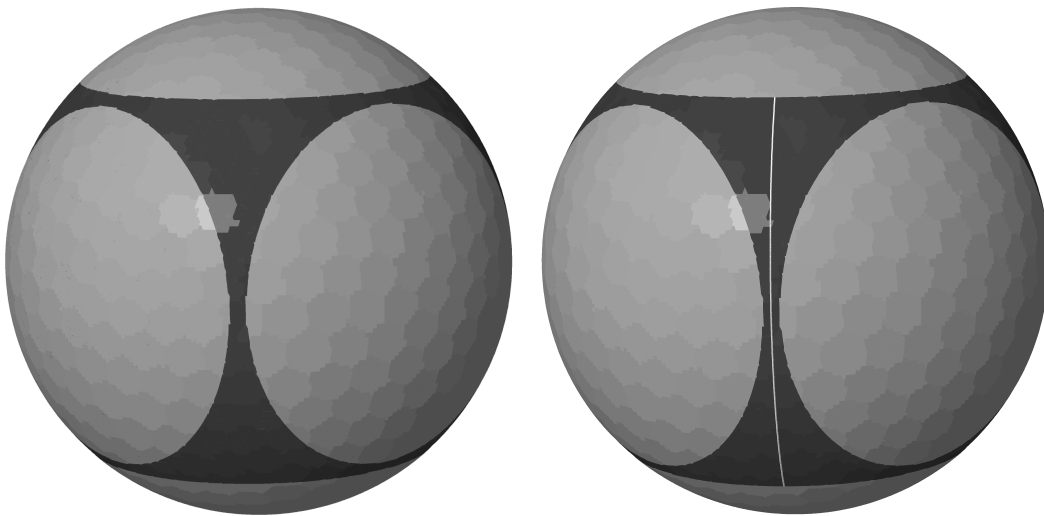


Figure 2: The trajectory which is shown on the right-hand side is not hyperbolic.

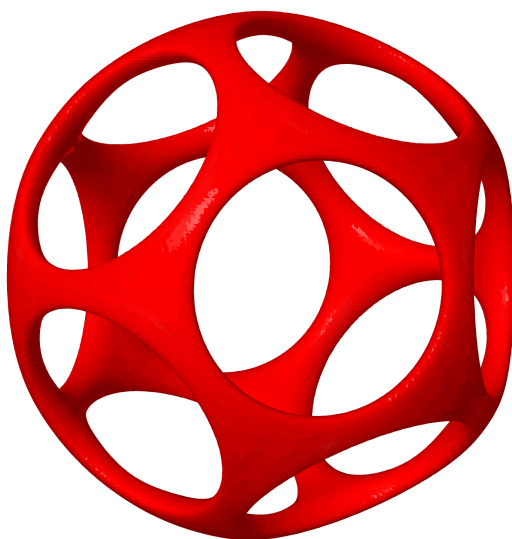


Figure 3: An approximation of the spherical billiard by a surface of genus 11. This surface is isometrically embedded in  $\mathbb{S}^3$  and is seen here in stereographic projection. Its geodesic flow is Anosov.

## 2 Main results

Consider a billiard  $D$  in the sphere  $\mathbb{S}^2$  with circular obstacles. Consider the largest obstacle and the length of its radius  $A$  (for the spherical metric), and the horizon  $H$  of the billiard. First we will prove:

**Theorem 2.1.** *If  $H < \pi/2$  and  $2 \tan(\pi/2 - A) > \tan(H)$ , then  $D$  is uniformly hyperbolic. In particular, it is the case if  $A + H < \pi/2$ .*

We will now see how uniform hyperbolicity “transfers” to surfaces which approximate such a billiard.

Consider the stereographic projection of  $\mathbb{S}^3$ , and the surface

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{S}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

For  $q \in \mathbb{R}^3$  denote by  $\rho(q)$  the radial unit vector at  $q$  (the unit vector which is positively colinear to the vector joining the origin to  $q$ ) and by  $\pi$  the natural projection of  $\mathbb{S}^3$  (minus the poles) onto  $\mathbb{S}^2$ :

$$\begin{aligned} \pi : \mathbb{R}^3 \setminus \{0\} &\rightarrow \mathbb{S}^2 \\ (x, y, z) &\rightarrow (x/(x^2 + y^2 + z^2), y/(x^2 + y^2 + z^2), z/(x^2 + y^2 + z^2)). \end{aligned}$$

We also consider the “flattening map”  $f_\epsilon$  for  $\epsilon \in (0, 1)$ :

$$\begin{aligned} f_\epsilon : \mathbb{R}^3 \setminus \{0\} &\rightarrow \mathbb{R}^3 \setminus \{0\} \\ q &\rightarrow \epsilon q + (1 - \epsilon)\pi(q). \end{aligned}$$

The main theorem of this paper is the following:

**Theorem 2.2.** *Consider a spherical billiard  $D$  with spherical obstacles  $\Delta_i$ , which satisfies  $H < \pi/2$  and  $2 \tan(\pi/2 - A) > \tan(H)$  (see the notations above), and a surface  $\Sigma$  in  $\mathbb{S}^3$  such that  $\pi(\Sigma) = D$ . We assume that:*

1. (Transversality to the fibers of the projection.) For all  $x \in \Sigma$ , if  $\pi(x) \notin \partial D$ , then  $\rho \notin T_x \Sigma$ .
2. (Nonzero vertical principal curvature.) For all  $x \in \Sigma$ , if  $\pi(x) \in \partial D$ , then  $\Pi_x(\rho) \neq 0$  (where  $\Pi$  is the second fundamental form).
3. (Symmetry.) For all  $i$ ,  $\partial \Delta_i \subseteq \Sigma$ ; moreover, there is a neighborhood of  $\partial \Delta_i$  in  $\Sigma$  which is invariant by inversion with respect to  $\mathbb{S}^2$ , and by rotation in  $\mathbb{S}^3$  around the axis  $(0, c_i)$ , where  $c_i$  is the center of  $\Delta_i$ .

Then there exists  $\epsilon_0 \in (0, 1)$  such that for all  $\epsilon \leq \epsilon_0$ , the geodesic flow on the flattened surface  $\Sigma_\epsilon = f_\epsilon(\Sigma)$  is Anosov.

The assumptions of Theorem 2.2 are very close to those which appear in the main theorem of [Kou16a] (which deals with the case of flat billiards), but the proof is made more difficult by the positive curvature of the sphere.

It is actually simple, from a given billiard, to construct a surface which satisfies these assumptions. More precisely:

**Theorem 2.3.** *If  $D$  is a spherical billiard with  $n$  circular obstacles, then there exists a surface  $\Sigma$  of genus  $n - 1$  such that  $\pi(\Sigma) = D$ , which satisfies the assumptions 1, 2 and 3 of Theorem 2.2.*

*Proof.* We assume that the circular obstacles have radii  $r_1, r_2, \dots, r_n$ . Choose  $\delta > 0$  such that the circles  $\tilde{\Delta}_i$  of radii  $r_i + \delta$ , with the same centers as the obstacles, remain disjoint. Consider the image  $S_1$  of  $\mathbb{S}^2 \setminus \bigcup_i \tilde{\Delta}_i$  by a homothety (in  $\mathbb{R}^3$ ) of center  $(0, 0, 0)$  and ratio  $1 - \epsilon$ , with a small  $\epsilon > 0$ . Consider the image  $S_2$  of  $S_1$  by inversion with respect to  $\mathbb{S}^2$ . Finally, construct symmetric tubes which connect the pairs of “holes” on the surfaces  $S_1$  and  $S_2$ .  $\square$

Theorem 2.3 will allow us to prove Theorem 1.2 in Section 8. On the other hand, we prove the following:

**Theorem 2.4.** *Consider a spherical billiard  $D$  with spherical obstacles  $\Delta_i$  of radii  $r_i < \pi/2$ , and a surface  $\Sigma$  in  $\mathbb{R}^3$  such that  $\pi(\Sigma) = D$  and  $\partial \Delta_i \subseteq \Sigma$ . Write  $r_{\max} = \max_i(r_i)$  and, for all  $\delta > 0$ , denote by  $V_i^\delta$  the open neighborhood of  $\partial \Delta_i$  in  $\Sigma$  which consists of all points at distance less than  $\delta$  from  $\partial \Delta_i$ . We will say that the surface  $\Sigma$  is  $\epsilon$ - $C^1$ -close to the billiard  $D$  if there exists locally a parametrization  $f_1$  of  $\Sigma$  and a parametrization  $f_2$  of  $D$  such that  $\|f_1 - f_2\|_{C^2} \leq \epsilon$  for the  $C^2$ -norm in the Euclidean  $\mathbb{R}^3$ .*

*For all  $\delta_1 \in (0, \pi/2 - r_{\max})$ , there exists  $\delta_2 > 0$ , such that the geodesic flow of any surface satisfying the following conditions has conjugate points:*

1. (Symmetry.) *The neighborhood  $V_i^{\delta_1}$  is invariant by rotation in  $\mathbb{R}^3$  around the axis  $(0, c_i)$ , where  $c_i$  is the center of  $\Delta_i$ .*
2. (Approximation of the billiard.) *The surface  $\Sigma \setminus \bigcup_i V_i^{\delta_2}$  is  $\delta_2$ - $C^2$ -close to  $D$ .*

In particular, for the surface  $\Sigma_\epsilon$  in Theorem 2.2, endowed with the metric induced by  $(\mathbb{R}^3, g_{\text{eucl}})$ , the geodesic flow always has conjugate points, and thus, it is never Anosov.

**Structure of the paper.** The proof of Theorem 2.1 relies on a theorem which is proved in [Kou16b]: in Section 3, we recall the statement of this theorem and finish the proof of Theorem 2.1. In Section 4, we state and prove a lemma in the Euclidean ambient space. This lemma is transposed to the spherical ambient space in Section 5. The local dynamics near a circular obstacle are studied in Section 6. Section 7 ends the proof of Theorem 2.2 by studying the global dynamics. We prove Theorem 1.2 in Section 8. Finally, we show Theorem 2.4 in Section 9.

### 3 The Riccati equation

The Riccati equation is an important tool for the proofs of this paper. On a smooth Riemannian manifold  $M$ , the Riccati equation is a differential equation along a geodesic  $\gamma : [a, b] \rightarrow M$  given by:

$$\dot{u}(t) = -K(t)u(t) - u(t)^2$$

where  $K$  is the Gaussian curvature of the surface.

The space of orthogonal Jacobi fields on the geodesic  $\gamma$  has dimension 1, thus it is possible to consider any orthogonal Jacobi field as a function  $j : [a, b] \rightarrow \mathbb{R}$  (by choosing an orientation of the normal bundle of  $\gamma$ ). In this case, it is well-known that  $j$  satisfies the

equation  $j''(t) = -K(t)j(t)$ . A short calculation then shows that  $u = j'/j$  satisfies the Riccati equation whenever  $j \neq 0$ .

The following criterion is an improvement of a statement which appears in [DP03]; a complete proof may be found in [Kou16b].

**Theorem 3.1.** *Let  $M$  be a closed surface. Assume that there exist  $m > 0$  and  $C > c > 0$  such that for any geodesic  $\gamma : \mathbb{R} \rightarrow M$ , there exists an increasing sequence of times  $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$  with  $c \leq t_{k+1} - t_k \leq C$ , such that the solution  $u$  of the Riccati equation with initial condition  $u(t_k) = 0$  is defined on the interval  $[t_k, t_{k+1}]$ , and  $u(t_{k+1}) > m$ . Then the geodesic flow  $\phi_t : T^1M \rightarrow T^1M$  is Anosov.*

Now, consider a spherical billiard  $D$  and a billiard trajectory  $\gamma$ . It is possible to consider a small variation of  $\gamma$ , which is also called a Jacobi field, and consider  $u = j'/j$ , as in the case of a geodesic flow. Thus, there is a natural generalization of the Riccati equation. We say that  $u$  is a solution of this equation if:

1. in the interval between two collisions,  $\dot{u}(t) = -K(t) - u(t)^2$ ;
2. when the particle bounces against the boundary at a time  $t$ ,  $u$  undergoes a discontinuity: we have  $u(t^+) = u(t^-) - \frac{2\kappa}{\sin \theta}$ , where  $\kappa$  is the geodesic curvature of the boundary of  $D$ , and  $\theta$  is the angle of incidence<sup>1</sup>.

With this generalized Riccati equation, the following theorem holds (its proof may be found in [Kou16b]):

**Theorem 3.2.** *Consider a spherical billiard  $D$ . Assume that there exist positive constants  $A, m, c$  and  $C$  such that for any trajectory  $\gamma$  with  $\gamma(0) \in \tilde{\Omega}$ , there exists an increasing sequence of times  $(t_k)_{k \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$  satisfying  $c \leq t_{k+1} - t_k \leq C$ , such that for any  $k \in \mathbb{Z}$ , the solution  $u$  of the Riccati equation with initial condition  $u(t_k^+) = 0$  satisfies  $u(t^+) \geq -A$  for all  $t \in [t_k, t_{k+1}]$ , and  $u(t_{k+1}^+) > m$ . Also assume that for each  $k \in \mathbb{Z}$ , there is no collision in the interval  $(t_k - c, t_k)$ , and at most one collision in the interval  $(t_k, t_{k+1}]$ . Then the billiard flow on  $D$  is uniformly hyperbolic (see Definition 1.3).*

Knowing this theorem, we are ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* Since the obstacles are circles of radius at most  $A$ , their geodesic curvature is at least  $\tan(\pi/2 - A)$ . We define  $m = 2 \tan(\pi/2 - A) - \tan(H)$  (thus  $m > 0$ ).

Consider a billiard trajectory  $(q(t), p(t))_{t \in [0, 2H]}$  in the billiard  $D$ , with collision times  $(t_k)_{k \in \mathbb{Z}}$ . Fix  $k$  and consider the solution  $u$  of the Riccati equation along this trajectory with  $u(t_k) = 0$ . We want to show that  $u(t_{k+1}) > \delta$ .

On  $(t_k, t_{k+1})$ , the solution  $u$  satisfies  $u'(t) = -1 - u(t)^2$ , so  $u(t_{k+1}^-) = -\tan(t_{k+1} - t_k) \geq -\tan(H)$ .

Knowing that  $u(t_{k+1}^+) = u(t_{k+1}^-) - \frac{2\kappa}{\sin \theta}$ , we obtain  $u(t_{k+1}^+) \geq -\tan(H) + 2 \tan(\pi/2 - A) = m$ . According to Theorem 3.2, this concludes the proof.  $\square$

<sup>1</sup>The notation  $u(t^+)$  stands for  $\lim_{h \rightarrow 0, h > 0} u(t + h)$ , and likewise  $u(t^-) = \lim_{h \rightarrow 0, h < 0} u(t + h)$ .

## 4 Flattening a curve in the Euclidean plane

**Lemma 4.1.** *Consider a smooth curve  $c : (-a, a) \rightarrow \mathbb{R}^2$  with unit speed, and write its coordinates  $c(t) = (r(t), z(t))$ . Assume that  $0 \in (a, b)$ ,  $c(0) = R$ ,  $c'(0) = (0, 1)$ , and the curvature of  $c$  at 0 is nonzero. For all  $t \in (a, b)$ , also assume that  $c(t) \geq R$ ,  $r(-t) = r(t)$  and  $z(-t) = -z(t)$ . Consider the flattened curve  $c^\epsilon(t) = (r(t), \epsilon z(t))$  and its curvature  $k^\epsilon(t)$ . The unit tangent vector to  $c^\epsilon(t)$  is  $T^\epsilon(t) = (T_r^\epsilon(t), T_z^\epsilon(t))$ , and the normal vector is  $(T_z^\epsilon(t), -T_r^\epsilon(t))$ .*

*Then there exists  $m_0 > 0$  such that for all  $m \leq m_0$ , there exists  $\epsilon_0 > 0$  such that for all  $\epsilon \leq \epsilon_0$ , there exists  $t_c$  such that*

1. *for all  $t \in (0, t_c)$ ,  $T_z^\epsilon(t) \geq m$  and  $k^\epsilon(t) \geq m^4/\epsilon^2$ ,*
2. *for all  $t \in (t_c, m)$ ,  $T_z^\epsilon(t) \leq m$  and  $k^\epsilon(t) \geq 0$ .*

*Proof.* We will write  $T^\epsilon(t) = (\cos \alpha^\epsilon(t), \sin \alpha^\epsilon(t))$  and  $s^\epsilon$  a parametrization by arc length of  $c^\epsilon$  (such that  $s^\epsilon(0) = 0$  and  $ds^\epsilon/dt = \|dc^\epsilon/dt\|$ ).

Since the curvature of  $c$  at  $t = 0$  is nonzero, we may assume that the angle  $t \mapsto \alpha(t)$  is decreasing on the interval  $(-m, m)$  (reducing  $m$  if necessary). We may also assume that  $\alpha(t) \in (0, \pi)$ .

We have:

$$T^\epsilon(t) = \frac{(\cos(\alpha^1(t)), \epsilon \sin(\alpha^1(t)))}{\sqrt{\cos^2(\alpha^1(t)) + \epsilon^2 \sin^2(\alpha^1(t))}}.$$

Thus,

$$\tan(\alpha^\epsilon(t)) = \epsilon \tan \alpha^1(t) \tag{1}$$

for  $t \in (0, m)$ .

In particular,  $t \mapsto \alpha^\epsilon(t)$  is decreasing (thus  $k^\epsilon(t) \geq 0$ ), so for  $\epsilon$  small enough, there exists  $t_c \in (0, m)$  such that  $\sin \alpha^\epsilon(t_c) = m$ . Thus for all  $t \in (t_c, m)$ , we have  $T_z^\epsilon(t) \leq m$ , whereas for all  $t \in (0, t_c)$ , we have  $T_z^\epsilon(t) \geq m$ .

We will now show that  $T_z^\epsilon(t) \geq m$  implies  $k^\epsilon(t) \geq m^4/\epsilon^2$ , which will conclude the proof of the lemma.

After differentiation of (1), we obtain

$$\begin{aligned} \frac{d\alpha^\epsilon}{dt}(1 + \tan^2(\alpha^\epsilon(t))) &= \epsilon \frac{d\alpha^1}{dt} \frac{1}{\cos^2(\alpha^1(t))} \\ \frac{d\alpha^\epsilon}{dt}(1 + \epsilon^2 \tan^2(\alpha^1(t))) &= \epsilon \frac{d\alpha^1}{dt} \frac{1}{\cos^2(\alpha^1(t))} \\ \frac{d\alpha^\epsilon}{dt} &= \frac{d\alpha^1}{dt} \frac{\epsilon}{\cos^2(\alpha^1(t)) + \epsilon^2 \sin^2(\alpha^1(t))} \\ \frac{d\alpha^\epsilon}{dt} &= \frac{d\alpha^1}{dt} \frac{\epsilon}{(ds^\epsilon/dt)^2} \end{aligned}$$

Thus the curvature of  $c_\epsilon$  is

$$k^\epsilon(t) = \frac{d\alpha^\epsilon}{ds} = \frac{d\alpha^\epsilon}{dt} \cdot \frac{1}{dt/ds^\epsilon} = k^1(t) \cdot \epsilon \cdot \frac{1}{(dt/ds^\epsilon)^3}.$$



Assuming that  $T_z^\epsilon \geq m$ , we obtain

$$\frac{\epsilon \sin \alpha}{ds^\epsilon/dt} = T_z^\epsilon \geq m$$

and thus

$$\frac{ds^\epsilon}{dt} \leq \frac{\epsilon}{m}.$$

Finally,

$$k^\epsilon(t) \geq k^1(t) \cdot \frac{m^3}{\epsilon^2} \geq \frac{m^4}{\epsilon^2}.$$

□

## 5 Curvature in a flattened tube

In sections 5, 6 and 7, we choose constants  $\nu$ ,  $m$ ,  $\delta$ , and  $\epsilon$ , in the interval  $(0, 1)$ , such that:

1. These constants are sufficiently small: how small they need to be depends on the choice of the billiard  $D$  and the surface  $\Sigma$ .
2. These constants satisfy  $\nu \gg m \gg \delta \gg \epsilon$ . This means that the ratios  $m/\nu$ ,  $\delta/m$  and  $\epsilon/\delta$  are sufficiently small, again with respect to the choice of  $D$  and  $\Sigma$ . We even assume that the ratios  $m/\nu^{1000}$ ,  $\delta/m^{1000}$  and  $\epsilon/\delta^{1000}$  are small.

We consider these constants as fixed once and for all, to avoid adding in each statement a prefix such as “there exists  $\nu_0 > 0$ , such that for all  $\nu \leq \nu_0$ , there exists  $m_0 > 0$ , such that for all  $m \leq m_0 \dots$ ”.

In this section, we consider a circular obstacle  $\Delta_{i_0}$ , with center  $q_0 \in \mathbb{S}^2$ , and use the stereographic projection of  $\mathbb{S}^3$  with  $q_0$  as the south pole (that is,  $q_0$  has coordinates  $(0, 0, 0)$ ). We will use cylindric coordinates  $(r, \theta, z)$ . The circle  $\Delta_{i_0}$  is defined by the equation  $r = R$ ,  $z = 0$ , where  $R \in (0, 1)$ .

Consider  $\tilde{\mathcal{T}} = \{(r, \theta, z) \in \Sigma_\epsilon \mid r \leq R + \delta\}$ , and  $\mathcal{T}$  the connected component of  $\tilde{\mathcal{T}}$  which contains  $\partial \Delta_{i_0}$ . The “tube”  $\mathcal{T}$  is a surface of revolution (assumption 3 of Theorem 2.2), obtained by rotation of a curve  $\mathcal{S}$  around the  $z$ -axis.

More precisely, we define the curve  $\mathcal{S}$  as the intersection of  $\mathcal{T}$  with the half great sphere corresponding to the equation  $\theta = 0$ . It has an upper part described by the equation  $z = h(r)$ , and a lower part given by  $z = -h(r)$ . Here, the mapping  $h$  is nonnegative, defined continuously on the interval  $[R, R + \delta]$ , smooth on  $(R, R + \delta)$ , and such that  $h(R) = 0$  (see Figure 4).

We consider the Euclidean metric  $g_{\text{eucl}}$  and the metric of the stereographic projection

$$g_{\text{stereo}} = \xi^2 g_{\text{eucl}}, \quad \text{where} \quad \xi = \frac{2}{(1 + r^2 + z^2)}.$$

The Euclidean scalar product is denoted by  $\langle \cdot \mid \cdot \rangle = g_{\text{eucl}}(\cdot, \cdot)$ , and the Euclidean norm is  $\|\cdot\| = \sqrt{\langle \cdot \mid \cdot \rangle}$ . The Levi-Civita connection of  $g_{\text{stereo}}$  is written  $\nabla$ .

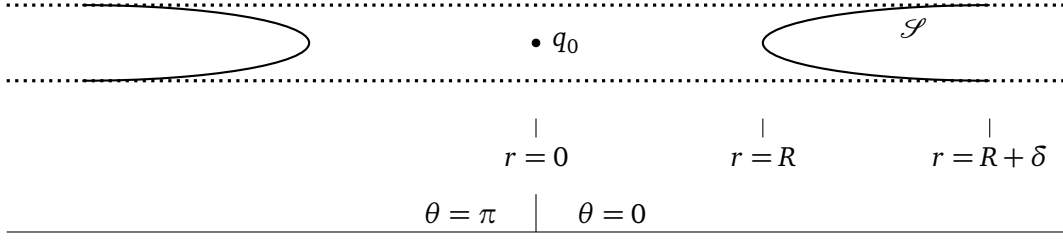


Figure 4: The curve  $\mathcal{S}$  (on the right-hand side) seen in stereographic projection. The two dotted lines correspond to two spheres in  $\mathbb{S}^3$  which are close to the great sphere  $\mathbb{S}^2$ .

There are three unit vectors  $e_r(q)$ ,  $e_\theta(q)$  and  $e_z(q)$  for each  $q = (x, y, z) \in \mathbb{R}^3 \setminus \{0\}$ , where (in cartesian coordinates)  $e_r(q) = (x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2}, 0)$ ,  $e_z = (0, 0, 1)$  and  $e_\theta = e_z \times e_r$ . The Euclidean norm of these vectors is 1. If  $p$  is a vector in  $\mathbb{R}^3 \setminus \{0\}$ , we will write  $p_r = \langle p \mid e_r \rangle$ ,  $p_\theta = \langle p \mid e_\theta \rangle$  and  $p_z = \langle p \mid e_z \rangle$ .

The second fundamental form of the surface  $\Sigma_\epsilon$  is defined by  $\Pi_q(v) = \xi^2 \langle \nabla_v v \mid N(q) \rangle$ , where  $N(q)$  is the unit normal vector to  $\Sigma_\epsilon$  for the metric  $g_{\text{stereo}}$  (thus  $\xi \|N\| = 1$ ). Any geodesic  $(q(t), p(t))$  on the tube satisfies the equation

$$\nabla_p p = \Pi_q(p)N(q).$$

When studying a tube, we always assume that  $N(q)$  points to the outside of the tube, and write  $N_r$ ,  $N_\theta$  and  $N_z$  its spherical coordinates.

By symmetry, at each point  $q$ , the two principal curvatures are  $k_1(q) = \Pi_q(e_\theta/\xi)$  and  $k_2 = \Pi_q(e_s/\xi)$ , where  $e_s$  is a unit vector which is orthogonal to  $e_\theta$  and tangent to  $\Sigma_\epsilon$ .

The curvature of  $\Sigma$  at  $q$  is  $k_1 k_2 + 1$ .

**Lemma 5.1.** *Consider the waist  $W$  of the tube  $\mathcal{T}$  (the smallest horizontal circle contained in the tube), and  $\kappa$  its curvature for the metric  $g_{\text{stereo}}$ . Then for all  $q = (q_r, q_\theta, q_z)$  in the tube:*

1.  $|k_1(q) - \xi N_r(q)\kappa| \leq m^2$ ;
2.  $k_1(q) \leq 0$ .

*Proof.* We consider a parametrization  $\mathcal{C}(t)$  of the horizontal circle contained in  $\Sigma$  such that  $\mathcal{C}(0) = q$ , with unit speed (for  $g_{\text{stereo}}$ ). The principal curvature  $k_1$  is

$$k_1 = \xi^2 \langle \nabla_{\mathcal{C}'(0)} \mathcal{C}'(0) \mid N(q) \rangle.$$

Since the circle  $\mathcal{C}$  is close to the circle  $W$ , we have

$$\left\| \nabla_{\mathcal{C}'(0)} \mathcal{C}'(0) - \kappa \frac{e_r}{\xi} \right\| \leq m^3$$

and thus

$$|k_1(q) - N_r(q)\kappa| = \left| \xi^2 \langle \nabla_{\mathcal{C}'(0)} \mathcal{C}'(0) \mid N(q) \rangle - \xi \kappa N_r(q) \right| \leq m^3 \xi^2 \leq m^2$$

which proves the first statement.

We now prove the second statement: by symmetry, we may assume that  $q_z \geq 0$ . Writing  $\nabla_{\mathcal{C}'(0)} \mathcal{C}'(0) = (r, \theta, z)$ , we obtain  $k_1 = \xi^2(rN_r(q) + zN_z(q))$ . Notice that  $r \leq 0$  and  $z \geq 0$ . At the same time,  $N_r(q) \geq 0$  and  $N_z(q) \leq 0$ . Thus,  $k_1 \leq 0$ .  $\square$

**Lemma 5.2.** *Consider a smooth curve  $q(t)$  in  $\Sigma_\epsilon$  and  $p(t) = \dot{q}(t)$ . Then:*

$$|\nabla_{p(t)} p(t) - \dot{p}(t)| \leq \|p(t)\|^2 / m^{1/10}.$$

*Proof.*

$$\nabla_{p(t)} p(t) = \sum_{1 \leq i, j, k \leq 3} p_i(t) p_j(t) \Gamma_{ij}^k e_k + \dot{p}(t),$$

where  $p_1, p_2, p_3$  mean respectively  $p_r, p_\theta$  and  $p_z$ , and  $e_1, e_2, e_3$  mean respectively  $e_r, e_\theta$  and  $e_z$ , and  $\Gamma_{ij}^k$  are the Christoffel symbols of  $\nabla$ .

Thus

$$|\nabla_{p(t)} p(t) - \dot{p}(t)| \leq \sum_{1 \leq i, j, k \leq 3} \|p(t)\|^2 |\Gamma_{ij}^k| \leq \|p(t)\|^2 / m^{1/10}.$$

$\square$

**Lemma 5.3.** *Consider the curve  $c : [a, b] \rightarrow \mathbb{R}^2$ ,  $c(t) = (r(t), z(t))$ , which is a unit-speed parametrization (for the metric  $g_{\text{stereo}}$ ) of the curve  $\mathcal{S}$ . Assume that  $0 \in (a, b)$ ,  $c(0) = R$  and  $c'(0) = (0, 1)$ . Recall that the curvature of  $c$  at 0 is nonzero. For all  $t \in (a, b)$ , we have  $r(t) \geq R$ ,  $r(-t) = r(t)$  and  $z(-t) = -z(t)$ . The unit tangent vector to  $c(t)$  is  $T(t) = (T_r(t), T_z(t))$  (that is,  $g_{\text{eucl}}(T, T) = 1$ ), and the normal vector is  $N = (T_z(t), -T_r(t))$ . The curvature of  $c$  for the metric  $g_{\text{stereo}}$  is written  $k_{\text{stereo}}$ .*

*Then there exists  $t_c$  such that*

1. *for all  $t \in (0, t_c)$ ,  $T_z(t) \geq m/2$  and  $k_{\text{stereo}}(t) \geq m^5/\epsilon^2$ ,*
2. *for all  $t \in (t_c, m)$ ,  $T_z(t) \leq 2m$  and  $k_{\text{stereo}}(t) \geq -1/m^{1/4}$ .*
3.  $\int_a^b k_{\text{stereo}}(t) dt \leq 1/m^{1/8}.$

*In the following, we will write  $r_c = r(t_c)$ .*

*Proof.* The strategy is to reduce the problem to a flat one, and then apply Lemma 4.1.

Consider the mapping  $\Phi : (0, 1) \times (-\sqrt{\epsilon}, \sqrt{\epsilon}) \rightarrow \mathbb{R}^2$ , whose restriction to the line  $z = 0$  is the identity, which is a bijection onto its image, and such that  $\Phi^{-1} \circ f_\epsilon \circ \Phi$  coincides with the affine map  $A : (r, z) \mapsto (r, \epsilon z)$  on its domain.

Then

$$\sup_{\mathbb{R} \times (-\sqrt{\epsilon}, \sqrt{\epsilon})} \|\Phi - \text{Id}\| < m^2 \quad \text{and} \quad \sup_{\mathbb{R} \times (-\sqrt{\epsilon}, \sqrt{\epsilon})} \|D\Phi - D\text{Id}\| < m^2$$

(“ $\Phi$  is  $C^1$ -close to the identity”). We will write  $\tilde{c}(t) = \Phi^{-1} \circ c(t)$ . Notice that  $\tilde{c}$  is a “flattened curve” in the Euclidean sense, and thus, Lemma 4.1 applies to  $\tilde{c}$ .

We denote by  $s$  the arc length of  $c$  for the metric  $g_{\text{eucl}}$ , and consider the curvature  $k_{\text{eucl}}$  of  $c$  for the metric  $g_{\text{eucl}}$ . Similarly, we denote by  $\tilde{s}$  the arc length of  $\tilde{c}$  for the metric  $g_{\text{eucl}}$ , and

consider the curvature  $\tilde{k}_{\text{eucl}}$  of  $\tilde{c}$  for the metric  $g_{\text{eucl}}$ . Also,  $\tilde{T}$  (*resp.*  $\tilde{N}$ ) is the unit tangent (*resp.* normal) vector to  $\tilde{c}$  for the metric  $g_{\text{eucl}}$ . Then:

$$\begin{aligned}
k_{\text{eucl}} &= \left\langle \frac{d \left( D\Phi(\tilde{c}(t)) \cdot \frac{\tilde{T}}{\|D\Phi(\tilde{c}(t)) \cdot \tilde{T}\|} \right)}{d\tilde{s}} \cdot \frac{d\tilde{s}}{ds} \middle| N \right\rangle \\
&= \left\langle D^2\Phi(c(t)) \cdot \left( \frac{\tilde{T}}{\|D\Phi(\tilde{c}(t)) \cdot \tilde{T}\|}, \tilde{T} \right) \cdot \frac{d\tilde{s}}{ds} \middle| N \right\rangle \\
&+ \left\langle D\Phi(\tilde{c}(t)) \cdot \left( \frac{d\tilde{T}}{d\tilde{s}} \cdot \frac{1}{\|D\Phi(\tilde{c}(t)) \cdot \tilde{T}\|} - \tilde{T} \frac{d}{ds} \left( \frac{1}{\|D\Phi(\tilde{c}(t)) \cdot \tilde{T}\|} \right) \right) \cdot \frac{d\tilde{s}}{ds} \middle| N \right\rangle \\
&\geq m^{1/10} \tilde{k}_{\text{eucl}}(t) - \frac{1}{m^{1/10}}.
\end{aligned}$$

On the other hand, using Lemma 5.2:

$$\begin{aligned}
\left| k_{\text{stereo}} - \frac{1}{\xi} k_{\text{eucl}} \right| &= \left| \xi^2 \left\langle \nabla_{T/\xi} T / \xi \middle| N / \xi \right\rangle - \frac{1}{\xi} \left\langle \frac{dT}{ds} \middle| N \right\rangle \right| \\
&= \left| \frac{1}{\xi} \left\langle \nabla_T T - \frac{dT}{ds} \middle| N \right\rangle \right| \\
&\leq \frac{1}{\xi m^{1/10}}.
\end{aligned}$$

Hence

$$k_{\text{stereo}}(t) \geq m^{1/9} \cdot k_{\text{eucl}}(t) - \frac{1}{m^{1/9}}.$$

Finally,

$$k_{\text{stereo}}(t) \geq m^{1/4} \cdot \tilde{k}_{\text{eucl}}(t) - \frac{1}{m^{1/4}}$$

and thus Lemma 4.1 applied to the curve  $\tilde{c}$  allows us to prove Statements 1 and 2.

We now prove Statement 3. From the inequality

$$\left| k_{\text{stereo}} - \frac{1}{\xi} k_{\text{eucl}} \right| \leq \frac{1}{\xi m^{1/10}}$$

we also obtain

$$k_{\text{stereo}} \leq \frac{1}{\xi} k_{\text{eucl}} + \frac{1}{\xi m^{1/10}}$$

and thus

$$\begin{aligned}
\int_a^b k_{\text{stereo}}(t) dt &\leq \int_a^b \left( k_{\text{eucl}}(t) + \frac{1}{m^{1/10}} \right) \frac{ds}{dt} dt \\
&\leq \left( \int_a^b 2 \frac{d\alpha}{ds} ds + (b-a) \frac{1}{m^{1/9}} \right) \\
&\leq 4\pi + \frac{(b-a)}{m^{1/9}} \leq 1/m^{1/8}.
\end{aligned}$$

□

## 6 The dynamics in the tube

In this section, we consider a unit speed geodesic  $(q(t), p(t))$  in  $T^1\mathcal{T}$ , where  $q$  is the position and  $p$  is the speed. We will write  $(q_r, q_\theta, q_z)$  and  $(p_r, p_\theta, p_z)$  the cylindric coordinates of  $q$  and  $p$ . Also, we write  $p_s = \sqrt{p_r^2 + p_z^2}$  and define  $e_s$  as the unit vector such that  $p_s = \langle p \mid e_s \rangle$ .

The field  $re_\theta$  is a Killing field on  $\mathcal{T}$ . Thus, the quantity  $L = g(re_\theta, p) = \xi^2 r p_\theta$  is constant on each geodesic (this is the Clairaut first integral).

The geodesic starts on the boundary of the tube at  $t = t^{\text{in}}$  and exits at  $t = t^{\text{out}}$  (if the geodesic does not exit the tube, we write  $t^{\text{out}} = +\infty$ ).

**Lemma 6.1.** *For all  $t \in [t^{\text{in}}, t^{\text{out}}]$ , we have*

$$|p_s(t) - p_s(t^{\text{in}})| \leq m^{10}.$$

*Proof.* For all  $t$ ,

$$p_s^2 = 1/\xi^2 - p_\theta^2 = 1/\xi^2 - \frac{L^2}{\xi^4 r^2}.$$

The coordinate  $r$  varies between  $R$  and  $R + \delta$ . Moreover, knowing that  $z \leq \delta$  (with  $\epsilon$  sufficiently small), the quantity  $\xi$  varies between  $\frac{2}{1+(R+\delta)^2+\delta^2}$  and  $\frac{2}{1+R^2}$ . Thus the variation of  $p_s^2$  is less than  $m^{10}$ . □

**Lemma 6.2.** *Assume that  $|p_s(t^{\text{in}})| \geq m$ . Then the time spent in the tube is smaller than  $6\delta/m$ .*

*Proof.* The length of the curve  $c^\epsilon$  is smaller than  $3\delta$ . Moreover, Lemma 6.1 implies that  $|p_s(t)| \geq m/2$  for all  $t$ . In particular,  $ds/dt$  does not change sign. Thus, the time spent in the tube is smaller than  $3\delta/(ds/dt) \leq 6\delta/m$ . □

**Lemma 6.3.** *Assume that  $|p_s(t^{\text{in}})| \geq m$ . Then*

$$\left| \int_{t^{\text{in}}}^{t^{\text{out}}} K(t) dt - \frac{2\kappa}{\xi(t^{\text{in}})p_s(t^{\text{in}})} \right| \leq m^{1/3}.$$

*Proof.* Let us divide the problem into several steps by using the triangle inequality (the integrals are taken between the times  $t^{\text{in}}$  and  $t^{\text{out}}$ ).

$$\begin{aligned}
& \left| \int_{t^{\text{in}}}^{t^{\text{out}}} K(t) dt - \frac{2\kappa}{\xi(t^{\text{in}})p_s(t^{\text{in}})} \right| \\
& \leq \left| \int K(t) dt - \int k_1 k_2 dt \right| + \left| \int k_1 k_2 dt - \int k_2 \xi N_r \kappa dt \right| \\
& + \left| \int k_2 \xi N_r \kappa dt - \int k_2 \xi \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\
& + \left| \int k_2 \xi \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt - \int \frac{1}{\xi(t)p_s(t)^2} \Pi_q(p) \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\
& + \left| \int \frac{1}{\xi(t)p_s(t)^2} \Pi_q(p) \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt - \int \frac{1}{\xi(t)p_s(t)^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\
& + \left| \int \frac{1}{\xi(t)p_s(t)^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt - \int \frac{1}{\xi(t^{\text{in}})p_s(t^{\text{in}})^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\
& + \left| \int \frac{1}{\xi(t^{\text{in}})p_s(t^{\text{in}})^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt - \frac{2\kappa}{\xi(t^{\text{in}})p_s(t^{\text{in}})} \right|.
\end{aligned}$$

We will now show that each of the terms is smaller than  $\sqrt{m}$ .

1. With Lemma 6.2,

$$\left| \int K(t) dt - \int k_1 k_2 dt \right| = \left| \int 1 dt \right| \leq 6\delta/m$$

2. With Lemma 5.1,

$$\left| \int k_1 k_2 dt - \int k_2 \xi N_r \kappa dt \right| \leq m \int k_2 dt = m \int \frac{k_2}{p_s} ds \leq 2 \int k_2 ds$$

Moreover, with Lemma 5.3,  $\int k_2 ds \leq 1/m^{1/8}$ , so

$$\left| \int K(t) - \int k_2 \xi N_r \kappa dt \right| \leq \sqrt{m}.$$

3. With Lemma 6.2, we know that  $e_r(t) - e_r(t^{\text{in}}) \leq \sqrt{\delta}$ . Thus:

$$\begin{aligned}
\left| \int k_2 \xi N_r \kappa dt - \int k_2 \xi \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| &= \left| \int k_2 \xi \langle N \mid e_r(t) - e_r(t^{\text{in}}) \rangle \kappa dt \right| \\
&\leq \delta^{1/4} \int k_2 ds \\
&\leq \sqrt{m}.
\end{aligned}$$

4. Using the fact that  $\Pi_q(p) = \xi^2(k_1 p_\theta^2 + k_2 p_s^2)$ , we obtain:

$$\begin{aligned} & \left| \int k_2 \xi \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt - \int \frac{1}{\xi(t)p_s(t)^2} \Pi_q(p) \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\ &= \left| \int k_1 \frac{p_\theta^2}{p_s^2} \xi \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\ &\leq \left| \int \xi \kappa (1+m) \frac{1}{\xi^2(m/2)^2} \xi \kappa dt \right| \leq \sqrt{m} \end{aligned}$$

5. With Lemma 5.2, since  $\Pi_q(p)N = \nabla_p p$ , we have:

$$\begin{aligned} & \left| \int \frac{1}{\xi(t)p_s(t)^2} \Pi_q(p) \langle N \mid e_r(t^{\text{in}}) \rangle \kappa dt - \int \frac{1}{\xi(t)p_s(t)^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\ &\leq \int \frac{\kappa}{\xi(m/2)^2 m} dt \leq \sqrt{m} \end{aligned}$$

6. We use Lemma 6.1:

$$\begin{aligned} & \left| \int \frac{1}{\xi(t)p_s(t)^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt - \int \frac{1}{\xi(t^{\text{in}})p_s(t^{\text{in}})^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt \right| \\ &\leq \left| m^{2/3} \int \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \right| \leq m^{2/3} \|p(t^{\text{out}}) - p(t^{\text{in}})\| \leq \sqrt{m} \end{aligned}$$

7. Since the tube is symmetric, we have  $\langle p_r(t^{\text{in}}) \mid e_r(t^{\text{in}}) \rangle = -\langle p_r(t^{\text{out}}) \mid e_r(t^{\text{out}}) \rangle$ .  
Thus:

$$\begin{aligned} & \left| \int \frac{1}{\xi(t^{\text{in}})p_s(t^{\text{in}})^2} \langle \dot{p} \mid e_r(t^{\text{in}}) \rangle \kappa dt - \frac{2\kappa}{\xi(t^{\text{in}})p_s(t^{\text{in}})} \right| \\ &= \left| \frac{1}{\xi(t^{\text{in}})p_s(t^{\text{in}})^2} \kappa \right| \left| \langle p(t^{\text{out}}) - p(t^{\text{in}}) \mid e_r(t^{\text{in}}) \rangle - 2 \langle p(t^{\text{in}}) \mid e_s(t^{\text{in}}) \rangle \right| \\ &\leq \frac{2}{m} \kappa m^2 \leq \sqrt{m}. \end{aligned}$$

□

In the following lemma, we consider the constant  $r_c$  given by Lemma 5.3.

**Lemma 6.4.** *In the tube, the time during which  $r \geq r_c$  is smaller than  $\delta^{1/3}$  (in other words,  $\int_{r \geq r_c} dt \leq \delta^{1/3}$ ).*

*Proof.* First, we compute for  $r \geq r_c$ :

$$\begin{aligned}
\frac{dr}{dt} &= p_r \\
&= \pm N_z \xi p_s \\
&= \pm N_z \xi \sqrt{\frac{1}{\xi^2} - p_\theta^2} \\
&= \pm N_z \xi \sqrt{\frac{1}{\xi^2} - \frac{L^2}{\xi^4 r^2}} \\
&= \pm N_z \sqrt{1 - L^2 f(r)}
\end{aligned}$$

where  $f(r) = \frac{(1+r^2+h(r)^2)^2}{4r^2}$ .

We compute:

$$\begin{aligned}
f'(r) &= 2(1+r^2+h(r)^2)(r^2-1+h(r)(2h'(r)-h(r)))/r^3 \\
&\leq -\delta^{1/10}.
\end{aligned}$$

Thus for all  $r_0 \geq r_c$  we may write:

$$f(r) \leq f(r_0) - \delta^{1/10}(r - r_0).$$

Moreover,  $f(r) \leq \delta^{-1/20}$ .

**First case.** We assume that there exists  $r_0 \geq r_c$  such that  $1 - L^2 f(r_0) = 0$ . Then the geodesic has the following life:  $r$  decreases from  $R + \delta$  to  $r_0$ , reaches  $r_0$  at some time  $t_0$ , and then increases from  $r_0$  to  $R + \delta$ . In this case, the length of the geodesic is

$$\begin{aligned}
2(t_0 - t^{\text{in}}) &= 2 \int_{t^{\text{in}}}^{t_0} dt \\
&= 2 \int_{r_0}^{R+\delta} \frac{1}{-dr/dt} dr \\
&= 2 \int_{r_0}^{R+\delta} \frac{1}{|N_z| \sqrt{1 - L^2 f(r)}} dr \\
&\leq 2 \int_{r_0}^{R+\delta} \frac{2}{\sqrt{1 - L^2 f(r_0) - L^2 \delta^{1/10}(r - r_0)}} dr \\
&\leq 4 \int_{r_0}^{R+\delta} \frac{1}{\sqrt{L^2 \delta^{1/10}(r - r_0)}} dr \\
&\leq \frac{8\sqrt{R+\delta-r_0}}{L\delta^{1/20}}
\end{aligned}$$



Since  $1 - L^2 f(r_0) = 0$ , we have  $L = 1/f(r_0)$  and thus  $L \geq \delta^{1/20}$ . Hence

$$2(t^{\text{out}} - t_0) \leq 8 \frac{\sqrt{\delta}}{\delta^{1/10}} \leq \delta^{1/3}.$$

**Second case.** We assume that  $1 - L^2 f(r) > 0$  for all  $r \in (r_c, R + \delta)$ . Then the geodesic goes through the zone  $r \geq r_c$  and enters the zone  $r \leq r_c$  at some time  $t_c$ . Then either it remains in this zone for all times, or it exits this zone and goes through the zone  $r \geq r_c$  once more. Thus, the time spent in the zone  $r \geq r_c$  is at most

$$\begin{aligned} 2 \int_{t^{\text{in}}}^t dt &= 2 \int_{r_c}^{R+\delta} \frac{1}{-dr/dt} dr \\ &= 2 \int_{r_0}^{R+\delta} \frac{1}{|N_z| \sqrt{1 - L^2 f(r)}} dr. \end{aligned}$$

If  $L \leq \delta^{1/20}$  then

$$2 \int_{t^{\text{in}}}^t dt \leq 4 \int_{r_c}^{R+\delta} \frac{1}{\sqrt{1 - \delta^{1/10} \delta^{-1/20}}} dr \leq \delta^{1/3}$$

which concludes the proof. Now assuming that  $L \geq \delta^{1/20}$ , we compute:

$$\begin{aligned} 2 \int_{t^{\text{in}}}^t dt &\leq 4 \int_{r_c}^{R+\delta} \frac{1}{\sqrt{1 - L^2 f(r_c) - L^2 \delta^{1/10} (r - r_c)}} dr \\ &\leq \frac{8\sqrt{R + \delta - r_c}}{L \delta^{1/20}} \\ &\leq \delta^{1/3} \end{aligned}$$

□

**Lemma 6.5.** *The Gauss curvature  $K$  of  $\Sigma_\epsilon$  satisfies  $K \leq \frac{1}{m^{1/3}}$ . Moreover,  $K \leq -1/\epsilon$  in the zone  $r \leq r_c$ .*

*Proof.* In the zone  $r \geq r_c$ , we have  $k_1 \leq 0$  (Lemma 5.1) and  $k_2 \geq -1/m^{1/4}$  (Lemma 5.3), so that  $K = k_1 k_2 + 1 \leq 1/m^{1/3}$ .

In the zone  $r \leq r_c$ , we have  $k_1 \leq -\kappa m/2$  (Lemmas 5.1 and 5.3) and  $k_2 \geq m^5/\epsilon^2$ . Thus, in this zone  $K = k_1 k_2 + 1 \leq -\kappa m^6/(2\epsilon^2) + 1$ . In particular,  $K \leq -1/\epsilon$ . □

**Lemma 6.6.** *In this lemma, we consider a geodesic  $(q(t), p(t))_{t \in [t^{\text{in}}, t^{\text{out}}]}$  in the tube  $\mathcal{T}$ , but we do not assume that  $q(t^{\text{in}})$  or  $q(t^{\text{out}})$  is on the boundary of  $\mathcal{T}$ . Consider a solution  $u$  of the Riccati equation  $u'(t) = -K(t) - u(t)^2$  such that  $|u(t^{\text{in}})| \leq 1/m^2$ . Then*

1.  $u(t^{\text{out}}) \geq u(t^{\text{in}}) - m$ ;

2. if the time spent in the tube  $\mathcal{T}$  is at least  $m$ , then  $u(t^{\text{out}}) \geq 1/m^2$ .

*Proof.* Let  $t^1 = \sup \{t \in [t^{\text{in}}, t^{\text{out}}] \mid u(t) \geq 2/m^2\}$  (if this set is empty, let  $t^1 = t^{\text{in}}$ ), and  $t^2 = \inf \{t \in [t^1, t^{\text{out}}] \mid u(t) \leq -2/m^2\}$  (if this set is empty, let  $t^2 = t^{\text{out}}$ ).

There is a (possibly empty) interval  $(t^3, t^4) \subseteq (t^1, t^{\text{out}})$  such that, for all  $t \in (t^1, t^{\text{out}})$ ,  $r(t) < r_c$  if and only if  $t \in (t^3, t^4)$ .

Assume that  $t^2 \leq t^3$ . Then using Lemmas 6.5 and 6.4, we have:

$$\begin{aligned} u(t^2) &= u(t^1) + \int_{t^1}^{t^2} -K(t) - u(t)^2 dt \\ &\geq u(t^1) - \frac{\delta^{1/3}}{m^{1/3}} - \frac{4\delta^{1/3}}{m^4} \\ &\geq u(t^1) - m/2 \end{aligned}$$

which contradicts the fact that  $u(t^2) \leq -2/m^2$ . Thus,  $t^2 \geq t^3$  and  $u(t^3) \geq u(t^1) - m/2$ .

For  $t \in (t^3, t^4)$ , we have:

$$u'(t) = -K(t) - u(t)^2 \geq \frac{1}{\epsilon} - 4/m^4 \geq 1/m^5$$

which implies that  $t^2 \geq t^4$  and  $u(t^4) \geq u(t^1) - m/2 + (t^4 - t^3)/m^5$ . Moreover,  $u(t^4) \leq 2/m^2$  (because  $t^1 \leq t^4$ ) so  $t^4 - t^3 \leq m^2$ . If  $t^{\text{out}} - t^{\text{in}} \geq m$ , this implies that  $t^1 > t^{\text{in}}$  and thus  $u(t^1) \geq 2/m^2$ .

Finally,

$$\begin{aligned} u(t^2) &= u(t^4) + \int_{t^4}^{t^2} -K(t) - u(t)^2 dt \\ &\geq u(t^1) - m/2 - \frac{\delta^{1/3}}{m^{1/3}} - \frac{4\delta^{1/3}}{m^4} \\ &\geq u(t^1) - m \end{aligned}$$

and thus  $t^2 = t^{\text{out}}$  and  $u(t^{\text{out}}) \geq u(t^1) - m \geq u(t^1) - m$ .

If the time spent in  $\mathcal{T}$  is at least  $m$ , then  $u(t^1) \geq 2/m^2$  and thus  $u(t^{\text{out}}) \geq 1/m^2$ .  $\square$

**Lemma 6.7.** Assume that  $|p_s(t^{\text{in}})| \geq m$ . Consider a solution  $u$  of the Riccati equation  $u'(t) = -K(t) - u(t)^2$  such that  $|u(t^{\text{in}})| \leq 2/m^2$ . Then:

$$\left| u(t^{\text{out}}) - u(t^{\text{in}}) - \frac{2\kappa}{\xi(t^{\text{in}})p_s(t^{\text{in}})} \right| \leq m^{1/4}.$$

*Proof.* Let  $t^1 = \inf \{t \in [t^{\text{in}}, t^{\text{out}}] \mid |u(t^{\text{in}})| \geq 2/m^2\}$  (if this set is empty, let  $t^1 = t^{\text{out}}$ ). We write  $K = K^+ - K^-$ , where  $K^+ = \max(K, 0)$  is the positive part of  $K$  and  $K^- = \max(-K, 0)$  is the negative part. Then, using Lemmas 6.5, 6.3 and 6.2,

$$\begin{aligned}
u(t^1) &= u(t^{\text{in}}) + \int_{t^{\text{in}}}^{t^1} -K(t) - u(t)^2 dt \\
|u(t^1)| &\leq |u(t^{\text{in}})| + \int_{t^{\text{in}}}^{t^{\text{out}}} |K(t)| + |u(t)|^2 dt \\
|u(t^1)| &\leq |u(t^{\text{in}})| + 2 \int_{t^{\text{in}}}^{t^{\text{out}}} K^+(t) dt - \int_{t^{\text{in}}}^{t^{\text{out}}} K(t) dt + \int_{t^{\text{in}}}^{t^{\text{out}}} |u(t)|^2 dt \\
&\leq \frac{1}{m^2} + 2 \cdot \frac{6\delta}{m} \cdot \frac{1}{m^{1/3}} + \frac{2|\kappa|}{m} + m^{1/3} + \frac{6\delta}{m} \cdot \frac{4}{m^4} \\
&< \frac{2}{m^2}
\end{aligned}$$

Thus  $t^1 = t^{\text{out}}$  and

$$\begin{aligned}
\left| u(t^{\text{out}}) - u(t^{\text{in}}) - \frac{2\kappa}{\xi(t^{\text{in}})p_s(t^{\text{in}})} \right| &= \left| u(t^{\text{out}}) - u(t^{\text{in}}) - \int_{t^{\text{in}}}^{t^{\text{out}}} K(t) dt \right| + \left| \int_{t^{\text{in}}}^{t^{\text{out}}} K(t) dt - \frac{2\kappa}{\xi(t^{\text{in}})p_s(t^{\text{in}})} \right| \\
&\leq \frac{6\delta}{m} \cdot \frac{4}{m^4} + m^{1/3} \\
&\leq m^{1/4}
\end{aligned}$$

□

## 7 End of the proof of Theorem 2.2

**Theorem 7.1.** We say that a curve  $\varphi : [a, b] \rightarrow D$  in the billiard  $D$  is “ $\eta$ -almost a geodesic” if

1.  $\|\phi'(t)\|_g \leq 1$  for all  $t \in [a, b]$ ,
2.  $d(\varphi(b), \varphi(a)) \geq b - a - \eta$ ,

where  $d$  is the Riemannian distance and  $g$  the Riemannian metric in the sphere  $\mathbb{S}^2$ .

Consider  $H$  the horizon of  $D$ . Then there exists  $\eta > 0$  such that for all  $\eta$ -almost geodesic  $\varphi$ ,  $a - b < H + \nu$ .

*Proof.* Assume that the conclusion is false. Then there exists a sequence  $\varphi_n$  of  $\frac{1}{n}$ -almost geodesics such that  $a = 0$  and  $b = H + \nu$ . By the Arzelà-Ascoli theorem, the sequence  $\varphi_n$  converges in the  $C^0$ -topology to a curve  $\varphi : [0, H + \nu] \rightarrow D$  which is a real geodesic in  $D$ . This contradicts the definition of  $H$ . □

*End of the proof of Theorem 2.2.* Consider a geodesic  $(q(t), p(t))_{t \in [0, H+2\nu]}$ .

During its lifetime, the geodesic enters and exits the tubes. We will say that the tube is *almost avoided* if the two following conditions are satisfied:

1.  $|p_s(t^{\text{in}})| \leq m$ ;

$$2. \ t^{\text{out}} - t^{\text{in}} \leq m.$$

**Lemma 7.2.** *If the geodesic almost avoids all the tubes in a time interval  $(t^1, t^2)$ , then  $t^2 - t^1 \leq H + \nu$ .*

*Proof.* The geodesic's projection  $\pi \circ q$  is  $\nu$ -almost a geodesic in  $\mathbb{S}^2$ , so we may apply Lemma 7.1.  $\square$

Now, consider an increasing sequence of times  $(t_k)_{k \in \mathbb{Z}}$  such that:

1. For each  $k \in \mathbb{Z}$ , either  $t_k$  is a time at which the geodesic exits a tube which is not *almost avoided* ("type A"), or  $t_k = t_{k-1} + H + 2\nu$  ("type B") ;
2. If the geodesic exits a tube which is not *almost avoided* at a time  $t^{\text{out}}$ , then there exists  $k \in \mathbb{Z}$  such that  $t_k = t^{\text{out}}$ .
3. For all  $k \in \mathbb{Z}$ ,  $\nu \leq t_{k+1} - t_k \leq H + 3\nu$ .

According to Theorem 3.1, we need to show that for any  $k \in \mathbb{Z}$  and any  $u$  solution of the Riccati equation along the geodesic  $(p(t), q(t))$  with initial condition  $u(t_k) = 0$ , the solution  $u$  is well-defined on  $[t_k, t_{k+1}]$  and  $u(t_{k+1}) > m$ .

In the sphere, since the curvature is 1, the geodesics follow the Riccati equation  $u'(t) = -1 - u(t)^2$ . If  $q(t)$  remains outside the tubes in the time interval  $(t^1, t^2)$ , since the metric on  $\Sigma_\epsilon$  is close to the metric of the Euclidean sphere, we have  $u(t^2) \geq \tan(\arctan(u(t^1)) - t^2) - \nu$ . This is also the case if one assumes that  $q(t)$  *almost avoids* all the tubes in this time interval (by Lemma 6.6).

**First case.** If  $t_k$  is of type A, then consider the first time  $t^{\text{in}}$  (with  $t^{\text{in}} \in [t_k, t_{k+1}]$ ) at which the geodesic enters a tube which is not *almost avoided* (such a time exists by Lemma 7.2). Then  $u(t^{\text{in}}) \geq -\tan(H + \nu) - \nu$ . Then, by Lemmas 6.6 and 6.7,

$$u(t_{k+1}) \geq u(t^{\text{in}}) + 2 \tan(\pi/2 - A) - \nu \geq -\tan(H + \nu) + 2 \tan(\pi/2 - A) - 2\nu \geq m.$$

**Second case.** If  $t_k$  is of type B, notice that  $q(t_k)$  is inside a tube which is not *almost avoided*, because of Lemma 7.2. Therefore, the geodesic remains in the tube during the interval  $[t_k + H + \nu, t_{k+1}]$ . Since  $t_{k+1} - (t_k + H + \nu) \geq \nu$ , we may apply Lemma 6.6 and obtain:  $u(t_{k+1}) \geq 1/m^2$ .

Thus Theorem 3.1 applies, and Theorem 2.2 is proved.  $\square$

## 8 Embedding surfaces of genus at least 11

Consider a billiard  $D_R$  obtained from 12 circles of equal radius  $R$  whose centers are the vertices of a icosahedron which is inscribed in  $\mathbb{S}^2$  (Figure 1). The circles are disjoint if and only if  $R < R_0 = \arctan(2)/2$ . The horizon  $H_R$  of the billiard  $D_R$  depends on  $R$ .

**Proposition 8.1.**

$$H_R \xrightarrow{R \rightarrow R_0} \pi - 2 \arctan(2).$$

*Proof.* Consider a sequence  $R_n$  such that  $R_n \xrightarrow{n \rightarrow \infty} R_0$ , and a sequence  $\gamma_n$  of portions of geodesics of  $\mathbb{S}^2$  of maximal length, which are contained in  $D_{R_n}$ . Then there is a subsequence of  $\gamma_n$  which converges uniformly to the portion of geodesic represented on Figure 5, whose length is  $\pi - 2 \arctan(2)$ .  $\square$

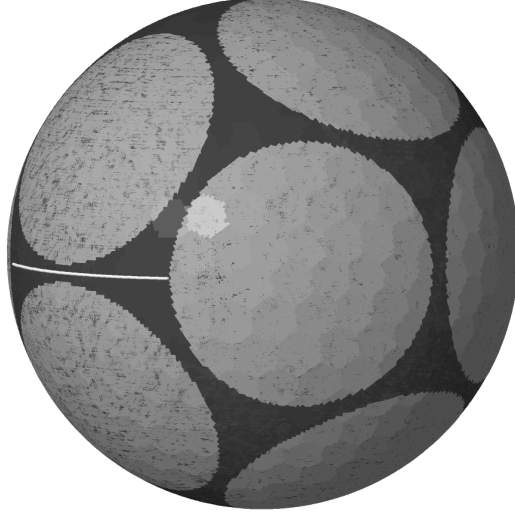


Figure 5: The limit of a sequence of geodesics of maximum lengths in  $D_{R_n}$ .

Thus

$$R + H_R \xrightarrow{R \rightarrow R_0} \pi - \frac{3}{2} \arctan(2) < \pi/2.$$

Therefore, for  $R$  sufficiently close to  $R_0$ , we have  $R + H_R < \pi/2$ , and thus  $2 \tan(\pi/2 - R) \geq \tan(\pi/2 - R) > \tan(H_R)$ .

By applying Theorem 2.1, we obtain

**Corollary 8.2.** *The billiard  $D$  is uniformly hyperbolic.*

To obtain a billiard with  $n$  obstacles ( $n \geq 12$ ), one may add spherical obstacles with small radii, which are disjoint from the others. Thus:

**Corollary 8.3.** *For any  $n \geq 12$ , there exists a spherical billiard with exactly  $n$  circular obstacles, which is uniformly hyperbolic.*

Finally, Theorem 2.3 completes the proof of Theorem 1.2.

## 9 The Euclidean case: proof of Theorem 2.4

We will study the geodesics in the “tube”  $V_{i_0}^{\delta_1}$  for some fixed  $i_0$ . We assume (after rotation) that the center  $q_0$  of the disk  $\Delta_{i_0}$  is on the  $z$ -axis. The tube is a surface of revolution in  $\mathbb{R}^3$ , obtained by rotation of a curve  $\gamma$  along the  $z$ -axis. We will use the cylindric coordinates  $(r, \theta, z)$ .

An essential difference with the spherical case is that the curve  $\gamma$  is *not* invariant by a symmetry with respect to a horizontal plane.

We assume that  $\gamma(s)$  is parametrized by arc length, that  $\gamma$  is in the half-plane  $\{\theta = 0\}$ , and that  $\gamma(0) \in \partial \Delta_{i_0}$ , with  $\gamma'(0) = e_z$ . Denote by  $\alpha(s)$  the angle of the tangent vector  $\gamma'(s)$  with the unit vector  $e_r$ , and consider the curvature  $k(s) = \frac{d\alpha}{ds}$ . Writing  $\gamma(s) = (\gamma_r(s), \gamma_z(s))$ , with the convention that the normal vector at  $\gamma(0)$  is  $e_r$ , the principal curvatures of the surface  $\Sigma$  at a point  $\gamma(s)$  are  $k_1 = -\frac{\cos(\alpha(s) - \pi/2)}{\gamma_r(s)}$  and  $k_2 = -k(s)$ ; thus the Gauss curvature is

$$K(s) = \frac{k(s) \cos(\alpha(s) - \pi/2)}{\gamma_r(s)} = \frac{\frac{d\alpha}{ds} \sin \alpha(s)}{\gamma_r(s)} = -\frac{d(\cos \alpha)/ds}{\gamma_r(s)}.$$

Assumption 2 of the theorem implies that  $K(s) \geq 0$  and  $\alpha(s) < -r_{i_0}/2$  for all  $s \in (\delta_2, \delta_1)$ . Moreover, since  $\alpha(0) = \pi/2$ , there exists  $\delta_3 \in (0, \delta_2)$  such that  $\alpha(\delta_3) = 0$ .

Consider a geodesic  $(p(t), q(t))$  in  $\Sigma$ , which enters the tube at a time  $-t_1$  and exits at time  $t_1$ . The symmetry assumption implies that the quantity  $L = \langle r(t)e_\theta(t) \mid p(t) \rangle = r(t)p_\theta(t)$  is constant on each unit speed geodesic. We assume that  $L = \gamma_r(\delta_3)$  (the intermediate value theorem guarantees the existence of such a geodesic). We will consider  $s(t)$  such that  $(r(t), z(t)) = \gamma(s(t))$ , and write  $K(s)$  the Gauss curvature at  $q(s(t))$ . We have  $p_s = \sqrt{1 - \frac{L^2}{r^2}}$ ; moreover,  $s(0) = \delta_3$  and there is a unique time  $t_2$  at which  $s(t_2) = \delta_2$ .

**Lemma 9.1.** *The following estimate holds:*

$$\int_0^{t_2} K(t) dt \geq \frac{1 - \cos(r_{i_0}/2)}{\sqrt{\gamma_r(\delta_2)^2 - \gamma_r(\delta_3)^2}}.$$

*Proof.* We compute:

$$\begin{aligned} \int_0^{t_2} K(t) dt &= \int_{\delta_3}^{\delta_2} \frac{K(s(t))}{ds/dt} ds \\ &= - \int_{\delta_3}^{\delta_2} \frac{d(\cos \alpha)/ds}{\sqrt{\gamma_r(s)^2 - \gamma_r(\delta_3)^2}} ds \\ &= \int_{\delta_3}^{\delta_2} g(s) \frac{d(\cos \alpha)}{ds} ds \end{aligned}$$

where

$$g(s) = -\frac{1}{\sqrt{\gamma_r(s)^2 - \gamma_r(\delta_3)^2}}.$$

Notice that  $g$  is differentiable on  $(\delta_3, \delta_2)$  with  $g'(s) > 0$ , and that there exists  $a > 0$  such that when  $s$  tends to  $\delta_3$ ,

$$g(s) = -\frac{a}{\sqrt{s - \delta_3}} + o\left(\frac{1}{\sqrt{s - \delta_3}}\right).$$

Now integrating by parts,

$$\int_{\delta_3}^{\delta_2} g(s) \frac{d(\cos \alpha)}{ds} ds = g(\delta_2)(\cos \alpha(\delta_2) - 1) - \lim_{s \rightarrow \delta_3} g(s)(\cos \alpha(s) - 1) - \int_{\delta_3}^{\delta_2} g'(s)(\cos \alpha(s) - 1) ds.$$

Since  $\lim_{s \rightarrow \delta_3} g(s)(\cos \alpha(s) - 1) = 0$ , and  $g'(s)(\cos \alpha(s) - 1) \leq 0$ , we obtain:

$$\begin{aligned} \int_0^{t_2} K(t) dt &\geq g(\delta_2)(\cos \alpha(\delta_2) - 1) \\ &\geq \frac{1 - \cos(r_{i_0}/2)}{\sqrt{\gamma_r(\delta_2)^2 - \gamma_r(\delta_3)^2}}. \end{aligned}$$

□

The existence of conjugate points in the geodesic flow is given by the following lemma:

**Lemma 9.2.** *Consider the Riccati equation along the geodesic  $(p(t), q(t))$ :*

$$u(0) = 0, \quad \frac{du}{dt} = -K(t) - u(t)^2.$$

*The solution of this equation blows up to  $-\infty$  in finite positive time, and to  $+\infty$  in finite negative time.*

*Proof.* First, notice that

$$u(t_2) \leq - \int_0^{t_2} K(t) dt \leq - \frac{1 - \cos(r_{i_0}/2)}{\sqrt{\gamma_r(\delta_2)^2 - \gamma_r(\delta_3)^2}}.$$

Moreover, for  $t \in (t_2, t_1)$ , since  $K(t) \geq 0$ , we have

$$\frac{du}{dt} \leq -u(t)^2$$

and so, for  $t \in (t_2, t_1)$ ,

$$u(t) \leq \frac{1}{t - t_2 + 1/u(t_2)}.$$

Thus, the solution of the Riccati equation blows up to  $-\infty$  before the time

$$t_2 - \frac{1}{u(t_2)} \leq t_2 + \frac{\sqrt{\gamma_r(\delta_2)^2 - \gamma_r(\delta_3)^2}}{1 - \cos(r_{i_0}/2)},$$

thus before the time  $t_1$ , provided that  $\delta_2$  is sufficiently small with respect to  $\delta_1$ .

By symmetry, the solution blows up to  $+\infty$  in negative times, after the time  $-t_1$ . □

This ends the proof of Theorem 2.4.

## Acknowledgements

This work was partially supported by the European Research Council.

## References

- [Arn63] Vladimir Igorevich Arnol'd. Small denominators and problems of stability of motion in classical and celestial mechanics. *Russian Mathematical Surveys*, 18(6):85–191, 1963.
- [Bir27] George D Birkhoff. Dynamical systems. 1927.
- [CM06] Nikolai Chernov and Roberto Markarian. *Chaotic billiards*, volume 127 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.
- [DP03] V. J. Donnay and C Pugh. Anosov geodesic flows for embedded surfaces. *Asterisque*, 287:61–69, 2003.
- [DP04] Victor J. Donnay and Charles Pugh. Finite horizon Riemann structures and ergodicity. *Ergodic Theory Dynam. Systems*, 24(1):89–106, 2004.
- [Kou16a] Mickaël Kourganoff. Anosov geodesic flows, billiards and linkages. *Comm. Math. Phys.*, 344(3):831–856, 2016.
- [Kou16b] Mickaël Kourganoff. Uniform hyperbolicity in nonflat billiards. *arXiv: 1605.00290*, 2016.
- [Sin70] Ja. G. Sinaï. Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards. *Uspehi Mat. Nauk*, 25(2 (152)):141–192, 1970.